Part I

Accounting System Dynamics

Chapter 1

System Dynamics

This chapter[1] introduces system dynamics from a dynamics viewpoint for beginners who have no formal mathematical background. First, dynamics is dealt in terms of a stock-flow relation. Under this analysis, a concept of DT (delta time) and differential equation is introduced together with Runge-Kutta methods. Secondly, in relation with a stock-dependent flow, positive and negative feedbacks are discussed. Then, fundamental behaviors in system dynamics are introduced step by step with one stock and two stocks. Finally, chaotic behavior is explored with three stocks, followed by discrete chaos.

1.1 Language of System Dynamics

What is system dynamics? The method of system dynamics was first created by Prof. Jay Forrester, MIT, in 1950s to analyze complex behaviors in social sciences, specifically, in management, through computer simulations [16]. It literally means a methodology to analyze dynamic behaviors of system. What is system, then? According to Jay Forrester, a founder of this field, "A system means a grouping of parts that operate together for a common purpose [17], page 1-1." For instance, following are examples of system he gave:

- An automobile is a system of components that work together to provide transportation.
- Management is a system of people for allocating resources and regulating the activity of a business.
- A family is a system for living and raising children.

[1] This chapter is based on the paper presented at the 17th International Conference of the System Dynamics Society: *Systems Thinking for the Next Millennium*, New Zealand, July 20 - 23 ,1999; specifically in the session L7: Teaching, Thursday, July 22, 1.30 pm - 3.00 pm, chaired by Peter Galbraith.

According to Edward Deming, a founder of quality control, " A system is a network of interdependent components that work together to try to accomplish the aim of the system [9], p. 50."

Both definitions share similar ideas whose keywords are: interdependent parts or components, and common purpose or aim.

To describe the dynamics of system thus defined, Forrester created a language of system dynamics consisting of four building blocks: Stock, Flow,Variable and information Arrow. They are illustrated in Figure 1.1. Flow is always connected to Stock. Arrow connects Variable, Flow and Stock.

This is a very genius idea. To describe a system, no matter how big it is, all we need is four building blocks (or letters) and their simple grammar. Compared with this, 26 letters in English and 55 phonetic letters in Japanese are required for writing a sentence, paper, or a book. As an analogy, let us consider our body as an example of system consisting of about 30 thousand genes. Yet they are created by four building blocks of DNA.

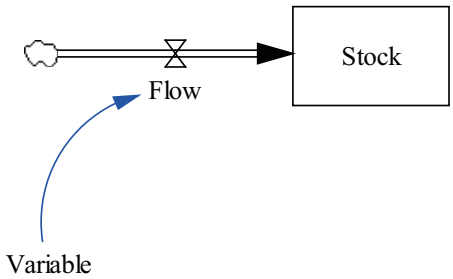


Figure 1.1: Language of System Dynamics

Textbooks and Softwares

Several textbooks are now available
for learning how to build a system with the above four building blocks such as [72] and [81] for business and management modeling, and [55] and [15] for sustainable environment. The reader is strongly recommended to learn system dynamics with these textbooks. I regularly use [72], [81] and [55] in my MBA and policy classes.

Another way to learn quickly is start using SD softwares such as Stella, Vensim and PowerSim with manuals. For the modeling in this book, I have selected Vensim for two reasons; its graphical capability for creating a model and availability of its free version such as Vensim PLE and Vensim Model Reader. It is recommended that the reader runs the models attached to this book simultaneously.

Economics students might need a little bit more rigorous approach to modeling in relation with difference and differential equations, which is, however, not well covered in the above introductory textbooks, This is why I decided to add another introduction to system dynamics method in this chapter.

1.2 Dynamics

1.2.1 Time

For beginners system dynamics seems to be an analysis of systems in terms of feedback mechanisms and interdependent relations. In particular this is true when graphics-oriented softwares of system dynamics become available for PCs and Macs such as Stella, Vensim and PowerSim, enabling even introductory students to build a complicated dynamic model easily without knowing a mechanism of dynamics and differential equations behind the screen.

Accordingly, the analysis of dynamics itself has been de-emphasized in a learning process of system dynamics. Dynamic analysis, however, has to be a foundation of system dynamics, through which systems thinking will be more effectively learned. This is what I have experienced when I encountered system dynamics as a new research field.

Dynamic analysis needs to be dealt along with a flow of time; an irreversible flow of time. What is time, then? It is not an intention to answer this philosophically deep question. Instead, time is here simply represented as an one dimensional real number, with an origin as its initial starting point, that flows toward a positive direction of the coordinate.

In this representation of time, two different concepts can be considered. The first concept is to represent time as a moment of time or a point in time, denoted here as τ; that is, time is depicted as a real number such that τ= 1, 2, 3, ... The second one is to represent it as a period of time or an interval of time, denoted here as t, such that t = 1st, 2nd, 3rd, ..., or more loosely t = 1, 2, 3, ... (a source of confusion for beginners). Units of the period could be a second, a minute, an hour, a week, a month, a quarter, a year, a decade, a century, a millennium, etc., depending on the nature of the dynamics in question.

In system dynamics, these two concepts of time needs to be correctly distinguished, because stock and flow - the most fundamental concepts in system dynamics - need to be precisely defined in terms of either τ or t as discussed below.

1.2.2 Stock

Let us now consider four building blocks or letters of system dynamics language in detail. Among those letters, the most important letter is stock. In a sense, system could be described as a collection of stock. What is stock, then? It could be an object to be captured by freezing its movement imaginably by stopping a flow of time, or more symbolically by taking its still picture. The object that can be captured this way is termed as stock in system dynamics. That is, stock is the amount that exists at a specific point in time τ , or the amount that has been piled up or integrated up to that point in time.

Let x be such an amount of stock at a specific point in time τ . Then stock can be defined as $x(\tau)$ where τ can be any real number.

Stocks thus defined may be classified according to their different types of nature as follows.

Physical Stock	Non-Physical Stock
• Natural Stock • Capital Stock • Goods-in-Process and Use	• Information • Psychological Passion • Indexed Figures

Table 1.1: Classification of Stock

- Natural stock consists of those that exist in our natural environment such as the amount of water in a lake, number of trees and birds in a forest and world population.
- Capital stock is a manufactured means of production such as buildings, factories and machines that have been used to produce final goods.
- Goods-in process are those that are in a process of production, which are sometimes called intermediate goods, and goods-in-use are final products that have been used by consumers such as cars and computers.
- Information (and knowledge) is non-physical stock that is stored in various forms of media such as papers, books, videos, tapes, diskettes, CDs and DVDs.
- Psychological passion is emotional stock of human beings such as love, joy, happiness, hatred and anger that have been stored somewhere in our brain tissues.
- Indexed figures are specific forms of information stock that are (scientifically) defined to describe the nature of environment and human activities such as temperature, prices, deposits and sales values.

1.2.3 Stock-Flow Relation

Since Newton, it has been a challenge in classical mechanics to describe a change in stock. One of the methods widely employed is to capture the amount of stock at various discrete points in time, $\tau = 0, 1, 2, 3, \cdots$ and consider a change in stock at the next point as the amount of the stock at the present point and its increment between the present and next points; that is, τ and $\tau + 1$. Let us call such an interval of time between these two points a unit interval. The length of unit could be, as already mentioned above, a second, a minute, an hour, a day, a week, a month, a year, or whichever unit to be suitable for capturing the movement of the stock in question. Hence, a period of time t could be defined as a t-th unit interval or period, counting from the origin; that is, $\tau = 0$.

Flow is defined as an increment (or decrement) of stock during a unit interval, and denoted here by $f(t)$. Flow that can only be defined at each discrete period of time is called discrete flow.

It is important to note that flow defined in this way is the amount between two points in time or a unit interval, while stock is the amount at a specific point in time. In other words, τ which is used for defining stock implies a point in time and t which is used for defining flow means a t-th unit interval between a point in time τ and its next period $\tau + 1$.

In this way, any dynamic movement can be operatively understood in terms of stock and flow. This stock-flow relation becomes fundamental to a dynamic analysis. It is conceptually illustrated in Figure 1.2.

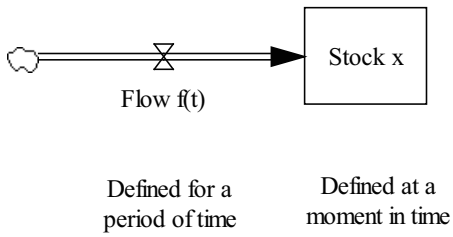


Figure 1.2: Stock-Flow Relation

It is essential to learn from the figure that flow is a part of stock, and in this sense physical or quantitative unit of flow and stock has to coincide. For instance, flow of oil cannot be added to the stock of water. As a system dynamics model becomes complicated, we tend to forget this essential fact.

Stock-flow relation can be formally written as

$$x(\tau + 1) = x(\tau) + f(t) \qquad \tau \text{ and } t = 0, 1, 2, 3, \cdots \tag{1.1}$$

To avoid a confusion derived from dual notations of time, τ and t, we need to describe stock-flow relation uniformly in terms of either one of these two concepts of time. Which one should, then, be adopted? A point in time τ could be interpreted as a limit point of an interval of time t. Hence, t can portray both concepts adequately, and can be chosen.

Since t represents a unit interval between τ and $\tau + 1$, the amount of stock at the t-th interval $x(t)$ could be defined as a balance at a beginning point τ of the period or an ending point $\tau + 1$ of the period; that is,

$$x(t) = x(\tau) : \text{Beginning balance of stock} \tag{1.2}$$

or

$$x(t) = x(\tau + 1) : \text{Ending balance of stock} \tag{1.3}$$

When the beginning balance of the stock equation (1.2) is applied, the stock-flow equation (1.1) becomes as follows:

$$x(t + 1) = x(t) + f(t) \qquad t = 0, 1, 2, 3, \cdots \tag{1.4}$$

In this formula, stock $x(t+1)$ is valued at the beginning of the period $t+1$; that is, flow $f(t)$ is added to the present stock value to give a stock value of the next period.

When the ending balance of the stock equation (1.3) is applied, the stock-flow equation (1.1) can be rewritten as

$$x(t) = x(t-1) + f(t) \qquad t = 1, 2, 3, \cdots \tag{1.5}$$

In this way, two different concepts of time - a point in time and a period of time - have been unified. It is very important for the beginners to understand that time in system dynamics always implies a period of time which has a unit interval. Of course, periods need not be discrete and can be continuous as well.

1.2.4 Integration of Flow

Discrete Sum

Without losing generality, let us assume from now on that $x(t)$ is an amount of stock at its beginning balance. If $f(t)$ is defined at a discrete time $t = 1, 2, 3, \ldots$, then the equation (1.4) is called a difference equation. In this case, the amount of stock at time t from the initial time 0 can be summed up or integrated in terms of discrete flow as follows:

$$x(t) = x(0) + \sum_{i=0}^{t-1} f(i) \tag{1.6}$$

This is a solution of the difference equation (1.4).

Continuous Sum

When flow is continuous and its measure at discrete periods does not precisely sum up the total amount of stock, a convention of approximation has been employed such that the amount of $f(t)$ is divided into n sub-periods (which is here defined as $\frac{1}{n} = \Delta t$) and n is extended to an infinity; that is, $\Delta t \to 0$. Then, the equation (1.4) can be rewritten as follows:

$$\begin{aligned} x(t) &= x\left(t - \frac{1}{n}\right) + \frac{f\left(t - \frac{1}{n}\right)}{n} \\ &= x(t - \Delta t) + f(t - \Delta t)\Delta t \end{aligned} \tag{1.7}$$

Let us further define

$$\lim_{\Delta t \to 0} \frac{x(t) - x(t - \Delta t)}{\Delta t} \equiv \frac{dx}{dt} \tag{1.8}$$

Then, for $\Delta t \to 0$ we have

$$\frac{dx}{dt} = f(t) \tag{1.9}$$

This formulation is nothing but a definition of differential equation. Continuous flow and stock are in this way transformed to differential equation, and the amount of stock at t is obtained by solving the differential equation. In other words, whenever a stock-flow diagram is drawn as in Figure 1.2, differential equation is constructed behind the screen in system dynamics.

The infinitesimal amount of flow that is added to stock at an instantaneously small period in time can be written as

$$dx = f(t)dt \tag{1.10}$$

Here, dt is technically called *delta time* or simply DT. Then an infinitesimal (or continuous) flow becomes a flow during a unit period t times DT.

Continuous sum is now written, in a similar fashion to a discrete sum in equation (1.40), as

$$x(t) = x(0) + \int_0^t f(u)du \tag{1.11}$$

This gives a general formula of a solution to the differential equation (1.9). The notational difference between continuous and discrete flow is that in a continuous case an integral sign is used instead of a summation sign. A continuous stock is, thus, alternatively called an integral function in mathematics.

In this way, stock can be described as a discrete or a continuous sum of flow[2]. This stock-flow relation becomes a foundation of dynamics (and, hence, system dynamics). It cannot be separable at all. Accordingly, among 4 letters of system dynamics language, stock-flow relation becomes an inseparable new building block or letter. Whenever stock is drawn, flows have to be connected to change the amount of stock. This is one of the most essential grammars in system dynamics.

1.3 Dynamics in Action

We are now in a position to analyze a dynamics of stock in terms of stock-flow relation. What we have to tackle here is how to find an efficient summation and integration method for different types of flow. Let us consider the most fundamental type of flow in the sense that it is not influenced (increased or decreased) by outside forces. In other words, this type of flow becomes autonomous and dependent only on time. Though this is the simplest type of flow, it is indeed worth being fully analyzed intensively by the beginners of system dynamics. Examples of this type to be considered here are the following:

- constant flow
- linear flow of time
- non-linear flow of time squared

[2]In Stella the amount of stock is described, similar to the equation (1.7), as

$$x(t) = x(t - dt) + f(t - dt) * dt,$$

while in Vensim it is denoted, similar to the equation (1.11), as

$$x(t) = INTEG(f(t),\ x(0)).$$

- random walk

Another examples would be trigonometric flow, present value, and time-series data, which are left uncovered in this book.

1.3.1 Constant Flow

The simplest example of this type of flow is a constant amount of flow through time. Let a be such a constant amount. Then the flow is written as

$$\boxed{f(t) = a} \tag{1.12}$$

This constant flow can be interpreted as discrete or continuous. A discrete interpretation of the stock-flow relation is described as

$$x(t+1) = x(t) + a \tag{1.13}$$

and a discrete sum of the stock at t is easily calculated as

$$x(t) = x(0) + at \tag{1.14}$$

On the other hand, a continuous interpretation of stock-flow relation is represented by the following differential equation:

$$\frac{dx}{dt} = a \tag{1.15}$$

and a continuous sum of the stock at t is obtained by solving the differential equation as

$$\begin{aligned} x(t) &= x(0) + \int_0^t a \; du \\ &= x(0) + at. \end{aligned} \tag{1.16}$$

From these results we can easily see that, if a flow is a constant amount through time, the amount of stock obtained either by discrete or continuous flow becomes the same.

1.3.2 Linear Flow of Time

We now consider autonomous flow that is linearly dependent on time. The simplest example of this type of flow is the following[3]:

[3]When time unit is a week, $f(t)$) has a unit of (Stock unit/ week) . Accordingly, in order to have the same unit, the right hand side has to be multiplied by a unitary variable of unit converter which has a unit of (Stock unit / week / week).

$$f(t) = t * \text{unit converter}$$

This process is called "unit check", and system dynamics requires this unit check rigorously to obtain equation consistency. In what follows in this introductory chapter, however, this unit check is not applied.

$$\boxed{f(t) = t} \tag{1.17}$$

Let the initial value of the stock be $x(0) = 0$. Then the analytical solution becomes as follows:

$$x(t) = \int_0^t u \, du = \frac{t^2}{2} \tag{1.18}$$

At the period $t = 10$, we have $x(10) = 50$. This is a true value of the stock. Stock and flow relation of the solution for $dt = 1$ is shown in Figure 1.4. The amount of stock at a time t is depicted as a height in the figure, which is equal to the area surrounded by the flow curve and time-coordinate up to the period t.

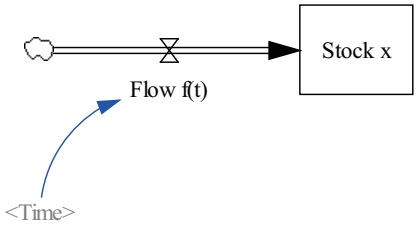


Figure 1.3: Linear Flow of Time

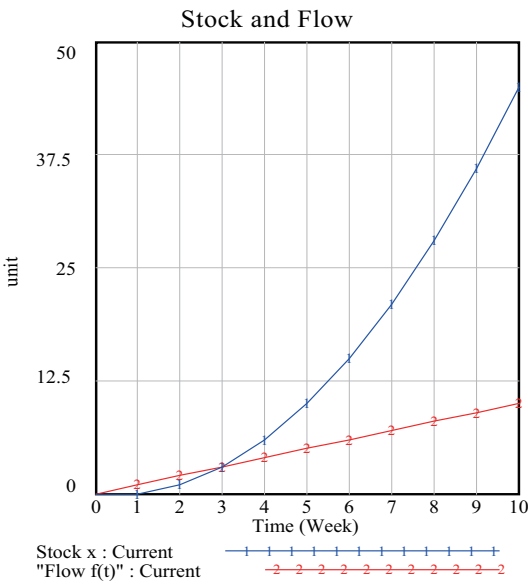


Figure 1.4: Linear Flow and Stock

Discrete Approximation

In general, the analytical (integral) solution of differential equation is very hard, or impossible, to obtain. The above example is a lucky exception. In such a general case, a numerical approximation is the only way to obtain a solution. This is done by dividing a continuous flow into discrete series of flow. Let us try to solve the above equation in this way, assuming that no analytical solution is possible in this case. Then, a discrete solution is obtained as

$$x(t) = \sum_{i=0}^{t-1} i \tag{1.19}$$

and we have $x(10) = 45$ at $t = 10$ with a shortage of 5 being incurred compared with a true value of 50. Surely, the analytical solution is a true solution. Only when it is hard to obtain it, a discrete approximation has to be resorted as an alternative method for acquiring a solution. This approximation is, however, far from a true value as our calculation shows.

a) Continuous Flow $dt \to 0$

Two algorithms have been posed to overcome this discrepancy. First algorithm is to make a discrete period of flow smaller; that is, $dt \to 0$, so that discrete flow appears to be as close as to continuous flow. This is a method employed in equation (1.7), which is known as the Euler's method. Table 1.2 shows calculations by the method for $dt = 1$ and 0.5. In Figure 1.4 a true value at $t = 10$ is shown to be equal to a triangle area surrounded by a linear flow and time-coordinate lines; that is, $10 * 10 * 1/2 = 50$. The Euler's method is, graphically speaking, to sum the areas of all rectangles created at each discrete period of time. Surely, the finer the rectangles, the closer we get to a true area.

Table 1.3 shows that as $dt \to 0$, the amount of stock gets closer to a true value of $x(10) = 50$, but it never gets to the true value. Meanwhile, the number of calculations and, hence, the calculation time increase as dt gets finer.

b) 2nd-Order Runge-Kutta Method

Second algorithm to approximate a true value is to obtain a better formula for calculating the amount of $f(t)$ over a period dt so that a rectangular area over the period dt becomes closer to a true area. The 2nd-order Runge-Kutta method is one such method. According to it, a value of $f(t)$ at the mid-point of dt is used.

$$x(t + dt) = x(t) + f(t + \frac{dt}{2})dt \tag{1.20}$$

Table 1.2: Linear Flow Calculation of $x(10)$ for dt=1 and 0.5

t	$x(t)$	dx
0	0	0
1	0	1
2	1	2
3	3	3
4	6	4
5	10	5
6	15	6
7	21	7
8	28	8
9	36	9
10	45	

where $dt = 1$ and $dx = f(t)dt = t$.

t	$x(t)$	dx	t	$x(t)$	dx
0.0	0.00	0.00	5.0	11.25	2.50
0.5	0.00	0.25	5.5	13.75	2.75
1.0	0.25	0.50	6.0	16.50	3.00
1.5	0.75	0.75	6.5	19.50	3.25
2.0	1.50	1.00	7.0	22.75	3.50
2.5	2.50	1.25	7.5	26.25	3.75
3.0	3.75	1.50	8.0	30.00	4.00
3.5	5.25	1.75	8.5	34.00	4.25
4.0	7.00	2.00	9.0	38.25	4.50
4.5	9.00	2.25	9.5	42.75	4.75
			10.0	47.50	

where $dt = 0.5$ and $dx = f(t)dt = 0.5t$.

Table 1.3: Discrete Approximation for $x(10)$

$f(t)\backslash dt$	1	1/2	1/4	1/8	1/16
Euler	45	47.5	48.75	49.375	49.6875
Runge-Kutta 2	50				

In our simple linear example here, it is calculated as

$$f(t + \frac{dt}{2}) = t + \frac{dt}{2} \tag{1.21}$$

Applying the 2nd-order Runge-Kutta method, we can obtain a true value even for $dt = 1$ as shown in Table 1.3.

1.3.3 Nonlinear Flow of Time Squared

We now consider non-linear continuous flow that is dependent only on time. The simplest example is the following:

$$\boxed{f(t) = t^2} \tag{1.22}$$

Let the initial value of stock be $x(0) = 0$. Then the analytical solution is obtained as follows:

$$x(t) = \int_0^t u^2 du = \frac{t^3}{3} \tag{1.23}$$

At the period $t = 6$, a true value of the stock becomes $x(6) = 72$. Figure 1.5 illustrates a stock and flow relation for this solution. Stock is shown as a height at a time t, which is equal to an area surrounded by a nonlinear flow curve and a time-coordinate up to the period t.

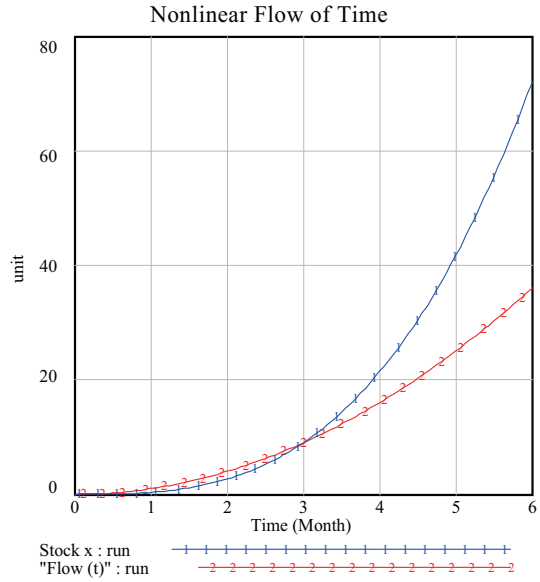


Figure 1.5: Nonlinear Flow and Stock

Discrete Approximation

A discrete approximation of this equation is obtained in terms of a stock-flow relation as follows:

$$x(t) = \sum_{i=0}^{t-1} i^2 \tag{1.24}$$

At the period $t = 6$, this approximation yields a value of 55, resulting in a large discrepancy of 17. In general, there exists no analytical solution for calculating a true value of the area surrounded by a nonlinear flow curve and a time-coordinate. Accordingly, two methods of approximation have been introduced; that is, the Euler's and 2nd-order Runge-Kutta methods. Table 1.4 shows such approximations by these methods for various values of dt. As shown in the table, both the Euler's and 2nd-order Runge-Kutta methods are not efficient to attain a true value for nonlinear flow even for smaller values of dt.

$$f(t + \frac{dt}{2}) = t^2 + tdt + \frac{dt^2}{4} \tag{1.25}$$

Clearly, in a case of nonlinear flow, the 2nd-order Runge-Kutta method, whether it be Stella or Madonna formula , fails to attain a true value at $t = 6$; that is, $x(6) = 72$.

Table 1.4: Discrete Approximation for $x(6)$

$f(t)\backslash dt$	1	1/2	1/4	1/8	1/16
Euler	55	63.25	67.5625	69.7656	70.8789
Runge-Kutta 2	71.5	71.875	71.9688	71.9922	71.998
Runge-Kutta 4	72.0				

4th-Order Runge-Kutta Method

The 4th-order Runge-Kutta method is a further revision to overcome the inefficiency of the 2nd-order Runge-Kutta method in a nonlinear case of flow as observed above. Its formula is given as[4]

$$x(t+dt) = x(t) + \frac{f(t) + 4f(t+\frac{dt}{2}) + f(t+dt)}{6}dt \tag{1.26}$$

Table 1.4 and 1.5 show that the 4th-order Runge-Kutta method are able to attain a true value even for $dt = 1$. The reader, however, should be reminded that this is not always the case as shown below.

Table 1.5: 2nd- and 4th-Order Runge-Kutta Method ($dt = 1$)

t	x(t)	Runge-Kutta 2	$x(t)$	Runge-Kutta 4
0	0	0.25	0	0.33
1	0.25	2.25	0.33	2.33
2	2.5	6.25	2.66	6.33
3	8.75	12.25	9	12.33
4	21	20.25	21.33	20.33
5	41.25	30.25	41.66	30.33
6	71.5		72	

1.3.4 Random Walk

Stochastic flow is created by probability distribution function. The simplest one is uniform random distribution in which random numbers are created between minimum and maximum. Let us consider the stock price whose initial value is \$ 10, and its price goes up and down randomly between the range of maximum \$1.00 and minimum - \$1.00.

$$\boxed{f(t) = \text{RANDOM UNIFORM (Minimum, Maximum)}} \tag{1.27}$$

Figure 1.6 is produced for a specific random walk. It is a surprise to see how a random price change daily produces a trend of stock price.

[4]Fore detailed explanation, see [11], section 2.8, pp.103 - 107 and 388 - 391.

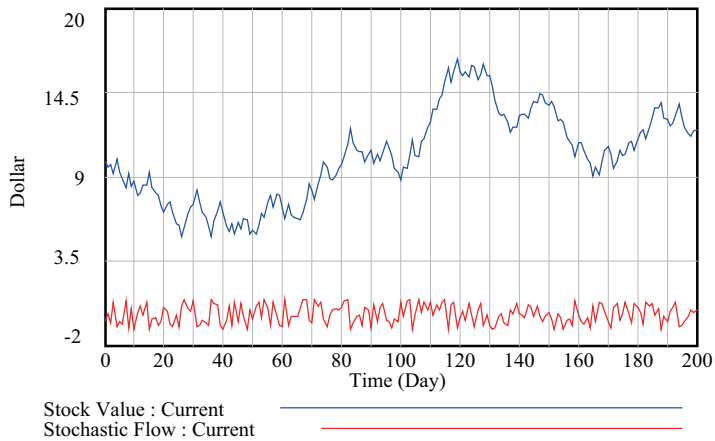


Figure 1.6: Random Walk

1.4 System Dynamics

1.4.1 Exponential Growth

So far the amount of flow is assumed to be created by autonomous outside forces at each period t. Next type of flow we now consider is the one caused by the amount of stock within the system. In other words, flow itself, being caused by the amount of stock, is causing a next amount of flow through a feedback process of stock: that is to say, flow becomes a function of stock. Whenever flow is affected by stock, dynamics becomes system dynamics.

When flow is discrete, a stock-flow relation of this feedback type is described as follows:

$$x(t+1) = x(t) + f(x(t)), \qquad t = 0, 1, 2, \cdots \tag{1.28}$$

In the case of a continuous flow, it is presented as a differential equation as follows.

$$\frac{dx}{dt} = f(x) \tag{1.29}$$

The simplest example of stock-dependent feedback flow is the following:

$$\boxed{f(x) = ax} \tag{1.30}$$

Figure 1.7 illustrates this stock-dependent feedback relation.

Its continuous flow is depicted as an autonomous differential equation:

$$\frac{dx}{dt} = ax \tag{1.31}$$

From calculus, an analytical solution of this equation is known as the following exponential equation:

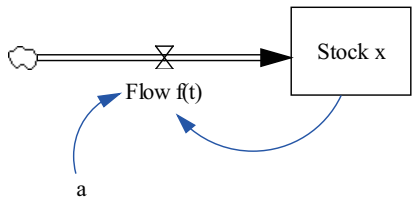


Figure 1.7: Stock-Dependent Feedback

$$x(t) = x(0)e^{at} \quad \text{where } e = 2.7182818284590452354\cdots \tag{1.32}$$

It should be noted that the initial value of the stock $x(0)$ cannot be zero, since non-zero amount of stock is always needed as an initial capital to launch a growth of flow.

What happens if such an analytical solution cannot be obtained? Assuming that flow is only discretely defined, we can approximate the equation as a discrete difference equation:

$$x(t+1) = x(t) + ax(t),\ t = 0, 1, 2, \cdots \tag{1.33}$$

Then, a discrete solution for this equation is easily obtained as

$$x(t) = x(0)(1+a)^t \tag{1.34}$$

A true continuous solution of the equation could be obtained as an approximation from this discrete solution (1.34), first by dividing a constant amount of flow a into n sub-periods, and secondly by making n sub-periods into infinitely many finer periods so that each sub-period converges to a moment in time.

$$\begin{aligned} x(t) &= x(0)\left[\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n\right]^t \\ &= x(0)\left[\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}}\right]^{at} \\ &= x(0)e^{at} \end{aligned} \tag{1.35}$$

Let an initial value of the stock be $x(0) = 100$ and $a = 0.1$. Then, a true value for the period $t = 10$ is $x(10) = 100e^{0.1*10} = 100e^1 = 271.8281828459\cdots$. This is to obtain a compounding increase in 10% for 10 periods. The amount of initial stock is shown to be increased by a factor of 2.7 when a growth rate is 10%. Table 1.6 shows numerical approximations by the Euler's, 2nd-order and 4th-order Runge-Kutta methods. It is observed from the table that even the 4th-order Runge-Kutta method cannot obtain a true value of the exponential e.

Table 1.6: Discrete Approximation

$f(t)\backslash dt$	1	1/2	1/4	1/8	1/16
Euler	259.374	265.33	268.506	270.148	270.984
Runge-Kuttta 2	271.408	271.7191	271.8004	271.8212	271.8264
Runge-Kutta 4	271.8279	271.828169	271.828182	271.828182	271.828182

In Figure 1.7, the amount of flow is shown to be determined by its previous amount through the amount of stock. Structurally this relation is sketched as a following flow chart: an increase in flow$\uparrow$ $\longrightarrow$ an increase in stock$\uparrow$ $\longrightarrow$ an increase in flow$\uparrow$. At an annual growth rate of 10%, for instance, it takes only seven years for the initial amount of stock to double, and 11 years to triple, and 23 years to become 10 folds. In fifty years, it becomes about 150 times as large. This self-increasing relation is called a *reinforcing or positive feedback* (see Figure 1.7.) Left-hand diagram in Figure 1.16 illustrates such a positive feedback growth.

Constant Doubling Times

One of the astonishing features of exponential growth is that a doubling time of stock is always constant. Let $x(0) = 1$ and $x(t) = 2$ in the equation (1.32), then it is obtained as follows.

$$ln2 = at \implies t = \frac{0.693147}{a} \tag{1.36}$$

For instance, when $a = 0.02$, that is, an annual growth rate is 2%, then doubling time of the stock becomes about 35 years. That is to say, every 35 years the stock becomes twice as big. When $a = 0.07$, or an annual growth rate is 7%, stock becomes doubled about every 10 years. Consider an economy growing at 7% annually. its GDP becomes 8 folds in 30 years. This enormous power of exponential is usually overlooked or under estimated. See the section of "Misperception of Exponential Growth" on pages 269 - 272 in [72].

Examples of Exponential Growth (Reinforcing Feedback)

In system dynamics, this exponential growth is called *positive* or *reinforcing* feedback. Figure 1.8 illustrates some examples of these reinforcing stock-dependent feedback relation. Left-hand diagram illustrates our financial system in which our bank deposits keeps increasing as long as positive interest rate is guaranteed by our banking system. This financial system creates the environment that "the rich becomes richer exponentially".

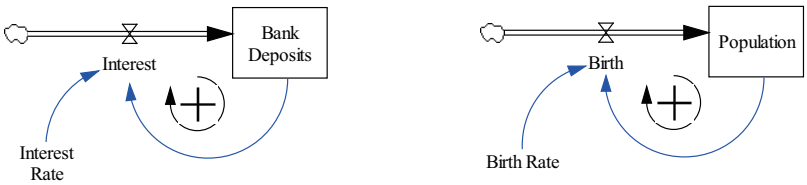


Figure 1.8: Examples of Exponential Feedback

1.4.2 Present Values

A frequently applied example of the exponential growth is observed in finance and economics to calculate present values of financial and real estate assets. From the equations (1.32) and (1.34), initial values of $x(0)$ increases exponentially to the values $x(t)$ at the period t. This implies that the initial values of $x(0)$ can be regarded as being equal to the intertemporal values $x(t)$. That is to say, future value of x(t) is only worth x(0) at the present (initial) time at $t = 0$.

More specifically, suppose $f(t)$ is an expected value of stream at the period t, $t = 0, 1, 2, \cdots$, and a is a discount rate of future, which can be represented by an interest rate or often referred to as marginal efficiency of capital in macroeconomics. Then, its present value $f(0)$ can be easily calculated from the equation (1.34) as

$$f(0) = \frac{f(t)}{(1+a)^t} \quad : \text{Present Value (Discrete)} \tag{1.37}$$

In the case of continuous time, its present value is calculated from the equation (1.32) as

$$f(0) = \frac{f(t)}{e^{at}} \quad : \text{Present Value (Continuous)} \tag{1.38}$$

Streams $f(t)$, $t = 0, 1, 2, \cdots$, are regarded as discount flows in system dynamics that constitute a stock of present values as illustrated in Figure 1.9. They could be payments of security interest, rentals from property and real estates, expected revenues of investment projects, and profits of corporate activities. Then, present values of securities, real estates, investment projects or corporations can be easily calculated, in the case of

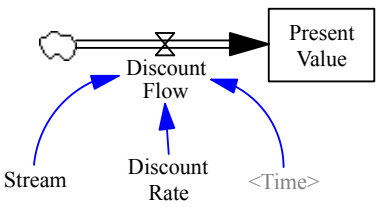


Figure 1.9: Present Value

discrete time, as follows:

$$\text{Present Value} = \sum_{t=0}^{\infty} \frac{f(t)}{(1+a)^t} \quad \text{(Discrete)}. \tag{1.39}$$

In the case of continuous time, they are obtained as

$$\text{Present Value} = \int_{t=0}^{\infty} \frac{f(t)}{e^{at}} dt \quad \text{(Continuous)}. \tag{1.40}$$

In this way, present value can be easily obtained, which is structurally the same as simple dynamics in Figure 1.3. Figure 1.10 compares two calculation of present values, discrete and continous. It is built with a constant value of stream (=100) and a discount rate of 10%. Line 3 and 4 indicates discrete and continuous discount flows, respectively, against which corresponding present values are calculated as line 1 and 2. Discrete present value slightly overestimates that of continuous value.

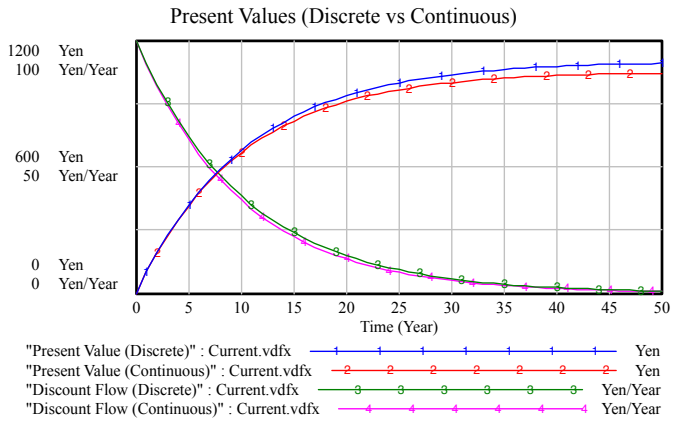


Figure 1.10: Present Values: Discrete vs Continuous

1.4.3 Balancing Feedback

If system consists of only exponential growth or reinforcing feedback behaviors, it will sooner or later explode. System has to have "a common purpose" or "the aim of the system" as already quoted in the beginning of this chapter. In other words, it has to be stabilized to accomplish its aim.

To attain a self-regulating stability of the system, another type of feedback is needed such that whenever a state of the system $x(t)$ is off the equilibrium x^*, it tries to come back to the equilibrium, as if it's being attracted to the

equilibrium. If system has this feature, it will be stabilized at the equilibrium. In economics, it is called global stability. Free market economy has to have this price stability as a system to avoid unstable price fluctuations.

Structurally the stability is attained if stock-flow has a relation such that an increase in flow↑ ⟶ a decrease in stock↓ ⟶ a decrease in flow↓. This stabilizing relation is called a *balancing or negative feedback* in system dynamics. Figure 1.11 illustrates this balancing feedback stock-dependent relation.

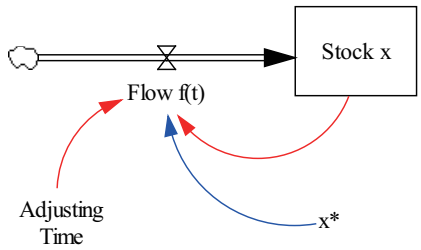


Figure 1.11: Balancing Feedback

Mathematically, this is to guarantee the stability of equilibrium. Let x^* be such an equilibrium point, or target or objective of the stock $x(t)$). Then stabilizing behavior is realized by the following flow

$$\boxed{f(x) = \frac{x^* - x(t)}{AT}} \tag{1.41}$$

where AT is the adjusting time of the gap between x^* and x.

Fortunately, this differential equation can be analytically solved as follows. First rewrite the equation (1.41) as

$$\frac{d(x(t) - x^*)}{x(t) - x^*} = -\frac{dt}{AT} \tag{1.42}$$

Then integrate both side to obtain

$$ln(x(t) - x^*) = -\frac{t}{AT} + C, \tag{1.43}$$

where ln denotes natural logarithm. This is further rewritten as

$$x(t) - x^* = e^{-\frac{t}{AT}} e^C \tag{1.44}$$

At the initial point in time, we have

$$e^C = x(0) - x^* \text{ at } t = 0 \tag{1.45}$$

Thus, the amount of stock at t is analytically obtained as

$$x(t) = x^* - (x^* - x(0))e^{-\frac{t}{AT}} \tag{1.46}$$

Examples of Balancing Feedback

Figure 1.12 illustrates some examples of these balancing stock-dependent feedback relation.

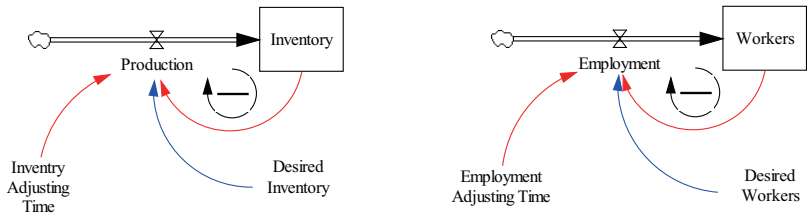


Figure 1.12: Examples of Balancing Feedback

Exponential Decay

When $x^* = 0$, system continues to decay or disappear. In other words, stock begins to decrease by the amount of its own divided by the adjusting time.

$$\boxed{f(x) = -\frac{x(t)}{AT}} \tag{1.47}$$

This decay process is called exponential decay. Whenever exponential decay appears, flow has only negative amount. In this case in system dynamics we draw flow out of stock so that it becomes more intuitive to understand the outflow of stock. Figure 1.13 illustrates such stock-outflow relation.

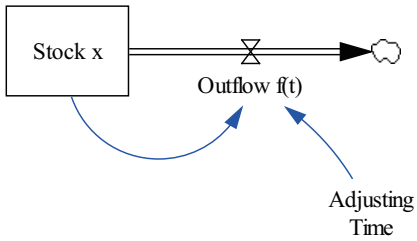


Figure 1.13: Exponential Decay

At an annual declining rate of 10%, for instance, the initial amount of stock decreases by half in seven years, by one third in 11 years, and by one tenth in 23 years, balancing to a zero level eventually. Right-hand diagram in Figure 1.16 illustrates such a negative feedback decay.

Examples of Exponential Decay

Figure 1.14 illustrates this stock-dependent feedback relation.

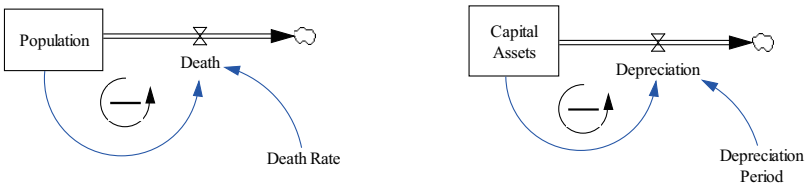


Figure 1.14: Examples of Exponential Decay

1.5 System Dynamics with One Stock

1.5.1 First-Order Linear Growth

We have now learned two fundamental feedbacks in system dynamics; reinforcing (exponential or positive) feedback and balancing (negative) feedback. Let us now consider the simplest system dynamics which have these two feedbacks simultaneously. It is called first-order linear growth system. "First-order" implies that the system has only one stock, while "linear" means that its inflow and outflow are linearly dependent on stock. Figure 1.15 illustrates our first system dynamics model which has both reinforcing and balancing feedback relations.

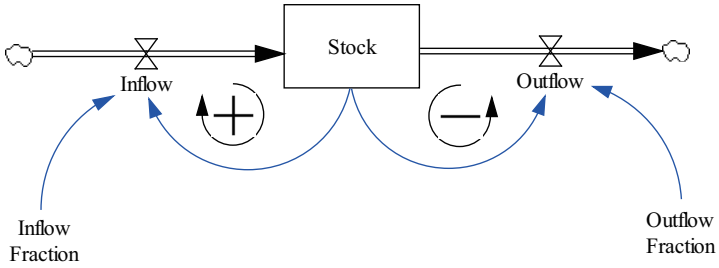


Figure 1.15: First-Order Linear Growth Model

Table 1.7 describes its equation.

Table 1.7: Equations of the First-Order Growth Model

Inflow= Stock*Inflow Fraction
Units: unit/Year
Inflow Fraction= 0.1
Units: 1/Year [0,1,0.01]
Outflow= Stock*Outflow Fraction
Units: unit/Year
Outflow Fraction= 0.04
Units: 1/Year [0,1,0.01]
Stock= INTEG (Inflow-Outflow, 100)
Units: unit

Left-hand diagram of Figure 1.16 is produced for the inflow fraction value of 0.1 and outflow fraction value of zero, and has a feature of exponential growth. Right-hand diagram is produced by the opposite fractional values, and has a feature of exponential decay. It is easily confirmed that whenever inflow fraction is greater than outflow fraction, the system produces exponential growth behavior. When outflow fraction is greater than inflow fraction, it causes an

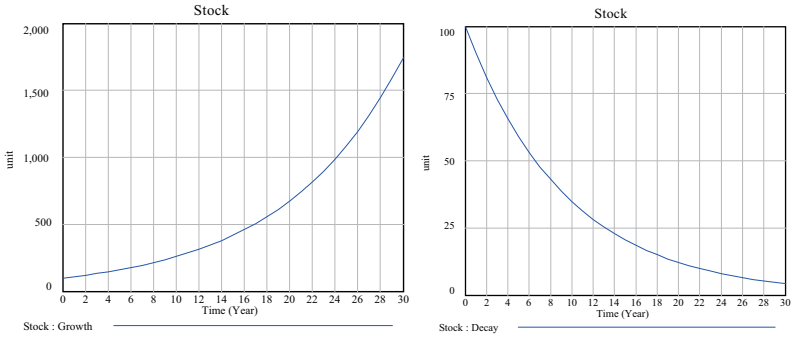


Figure 1.16: Exponential Growth and Decay

exponential decay behavior. In this way, the first-order linear system can only produces two types of behaviors: exponential growth or decay.

This model can best describe population dynamics. Suppose the world birth rate (inflow fraction) is 3.5%, while its death rate (outflow fraction) is 1.5%. This implies that world population grows exponentially at the net growth rate of 2%.

1.5.2 S-Shaped Limit to Growth

In the first-order linear model, the system may explode if inflow fraction is greater than outflow fraction. Population explosion is a good example. To stabilize the system, the exponential growth $ax(t)$ has to be curbed by bringing another balancing feedback which plays a role of a break in a car.

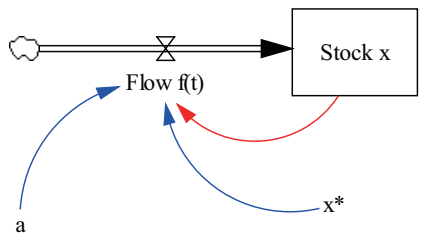


Figure 1.17: S-shaped Growth Model

Specifically, whenever $x(t)$ grows to a limit x^*, it begins to be regulated as if population is controlled and speed of the car is reduced. This is a feedback mechanism to stabilize the system. It could be done by the following flow:

$$\boxed{f(x) = ax(t)b(t), \text{ where } b(t) = \frac{x^* - x(t)}{x^*}} \tag{1.48}$$

Apparently $b(t)$ is bounded by $0 \leq b(t) \leq 1$, and reduces to zero as $x(t)$ approached to its limit x^*. Figure 1.17 is such model in which both reinforcing (exponential) and balancing feedback are brought together. It is called S-shaped growth in system dynamics. Figure 1.18 illustrates its behaviors.

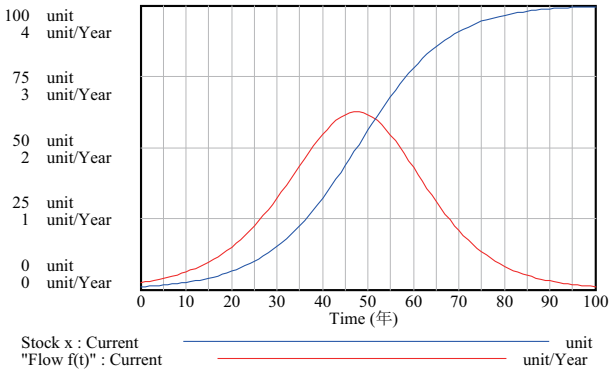


Figure 1.18: S-Shaped Limit to Growth 1

1.5.3 S-Shaped Limit to Growth with Table Function

Another way to regulate the growth of $x(t)$ is to increase $b(t)$ to the value of a as $x(t)$ grows to its limit x^* as shown below.

$$\boxed{f(x) = (a - b(t))x(t)} \tag{1.49}$$

Specifically, any functional relation that has a property such that $b(t)$ approaches a whenever $x(t)/x^*$ approaches 1 works for this purpose.

One of the simplest function is

$$b(t) = a\frac{x(t)}{x^*} \tag{1.50}$$

In this case the above function becomes

$$\boxed{f(x) = (a - b(t))x(t) = a\left(\frac{x^* - x(t)}{x^*}\right)x(t)} \tag{1.51}$$

which becomes the same as the above S-shaped limit to growth.

If mathematical function is not available, still we can produce S-shaped behavior by plotting the relation, which is called table function. One of such table function is shown in the right-hand diagram of Figure 1.19. Left-hand diagram illustrates S-shaped limit to growth model. Figure 1.20 illustrates its behaviors.

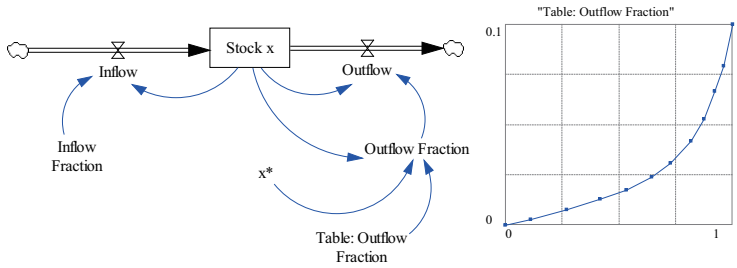


Figure 1.19: S-Shaped Limit to Growth Model with Table Function

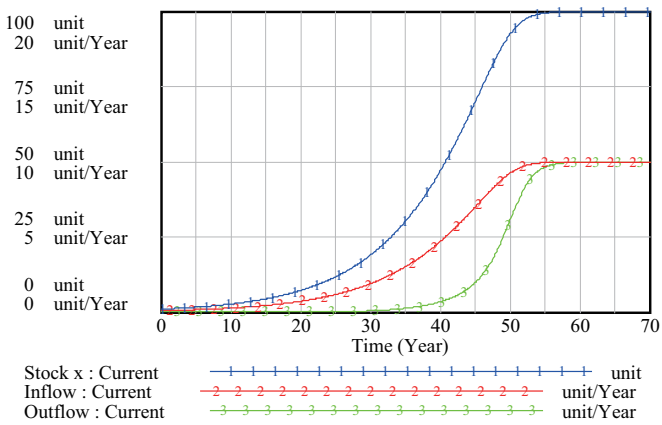


Figure 1.20: S-Shaped Limit to Growth 2 with Table Function

1.6 System Dynamics with Two Stocks

1.6.1 Feedback Loops in General

When there is only one stock, two feedback loops are at maximum produced as in first-order linear growth model. When the number of stocks becomes two, at maximum three feedback loops can be generated as illustrated in Figure 1.21

Mathematically, general feedback loop relation with two stocks can be represented by a following dynamical system in which each flow is a function of stocks x and y.

$$\frac{dx}{dt} = f(x, y) \tag{1.52}$$

$$\frac{dy}{dt} = g(x, y) \tag{1.53}$$

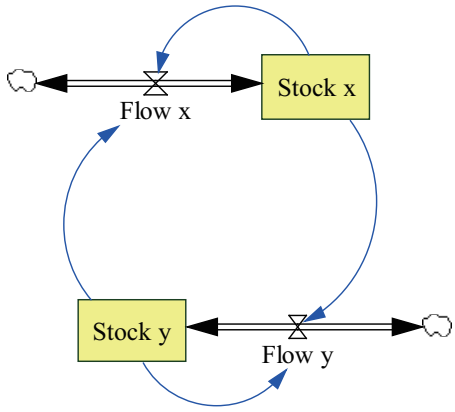


Figure 1.21: Feedback Loops in General

1.6.2 S-Shaped Limit to Growth with Two Stocks

Behaviors in system dynamics with one stock are limited to exponential growth and decay generated by the first-order linear growth model, and S-shaped limit to growth. To produce another fundamental behaviors such as overshoot and collapse, and oscillation, at least two stocks are needed. System dynamics with two stocks are called second-order system dynamics.

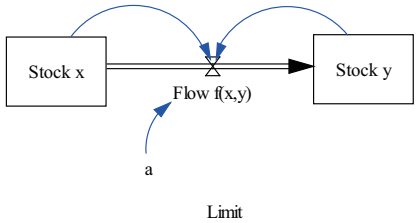


Figure 1.22: S-shaped Growth Model

Let us begin with another type of S-shaped limit to growth behavior that can be generated with two stocks x and y. When the total amount of stock x and stock y is limited by the constant available resources such that $x + y = b$, and the amount of stock x flows into stock y as shown in Figure 1.22, stock y begins to create a S-shaped limit to growth behavior.

A typical system causing this behavior is described as follows.

$$\frac{dx}{dt} = -f(x, y) \tag{1.54}$$

$$\begin{aligned} \frac{dy}{dt} &= f(x, y) \\ &= ax(t)y(t) \\ &= a(b - y(t))y(t) \end{aligned} \tag{1.55}$$

Figure 1.23 illustrates its behaviors for the values $a = 0.001$ and $b = 100$.

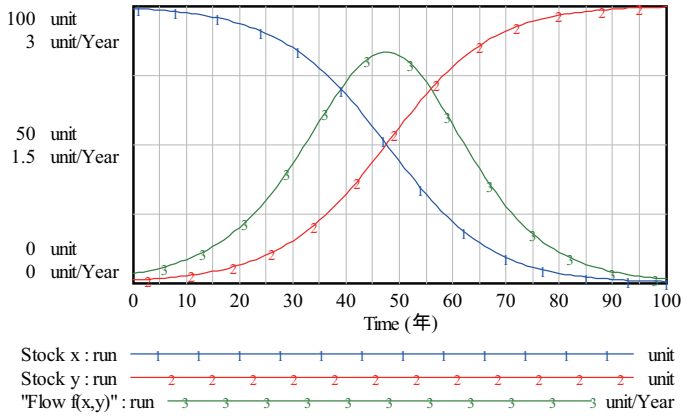


Figure 1.23: S-Shaped Limit to Growth 3

This relation is also reduced to

$$\frac{dx}{dt} + \frac{dy}{dt} = 0 \tag{1.56}$$

Mathematically, the above equation (1.55) is similar to the S-shaped limit to growth equation (1.48). In other words, $x(t) = b - y(t)$ begins to diminish as $y(t)$ continues to grow. This is a requirement to generate S-shaped limit to growth.

Examples of this type of S-shaped limit to growth are abundant such as logistic model of innovation diffusion in marketing.

So far we have presented three different figures to illustrate S-shaped limit to growth. Mathematically, all of the S-shaped limits to growth turn out to have the same structure. The same structures can be built in three different models, depending on the issues we want to analyze. This indicates the richness of system dynamics approach.

1.6.3 Overshoot and Collapse

Next behavior to be generated with two stocks is a so-called overshoot and collapse. It is basically caused by the S-shaped limit to growth model with table function. However, coefficient $b(t)$ is this time affected by the stock y. Increasing stock x causes stock y to decrease, which in turn makes the availability of stock y smaller, which then increases outflow fraction. This relation is described by the table function in the right-hand diagram of Figure 1.24. Increasing fraction collapses stock x. The model is shown in the left-hand diagram. Behaviors of overshoot & collapse is shown in Figure 1.25.

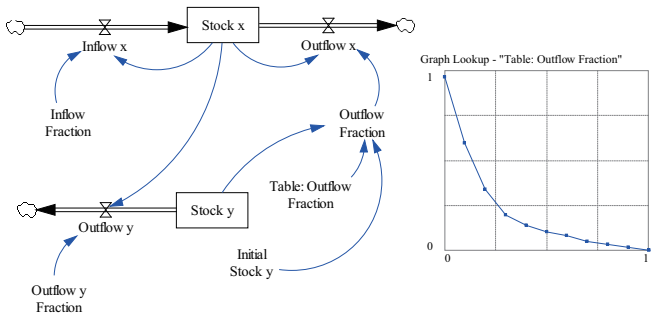


Figure 1.24: Overshoot & Collapse Model

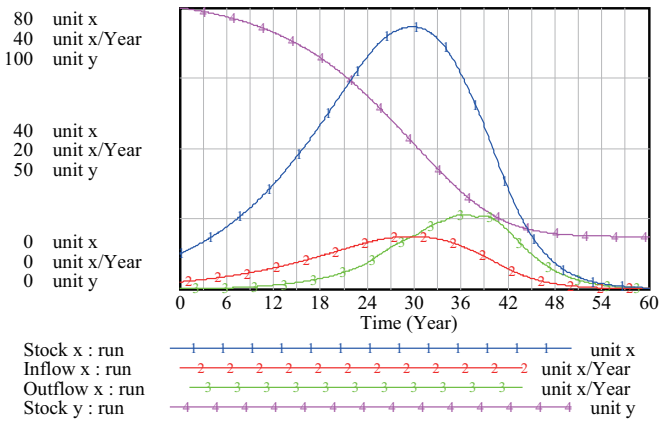


Figure 1.25: Overshoot & Collapse Behavior

Examples of Overshoot and Collapse

One of the favored examples of overshoot and collapse model is the decline of the Mayan empire in [4].

1.6.4 Oscillation

Another behavior that can be created with two stocks is oscillation. A simple example of system dynamics with two stocks is illustrated in Figure 1.26.

It can be formally represented as follows:

$$\frac{dx}{dt} = ay, \; x(0) = 1, \; a = 1 \tag{1.57}$$

$$\frac{dy}{dt} = -bx, \; y(0) = 1, \; b = 1 \tag{1.58}$$

This is nothing but a system of differential equations, which is also called a dynamical system in mathematics. Its solution by Euler method is illustrated in Figure 1.27 in which DT is set to be $dt = 0.125$.

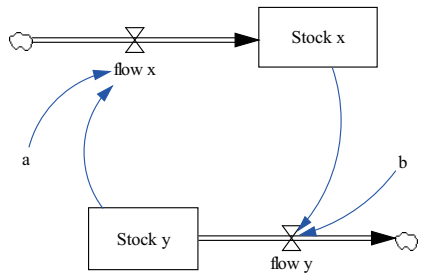


Figure 1.26: An Oscillation Model

Movement of the stock y is illustrated on the y-axis against the x-axis of time (left figure) and against the stock of x (right figure). In this Euler's solution, the amount of stock keeps expanding even a unit period is divided into 8 sub-periods for better computations. In this continuous case of flow, errors at each stage of calculation continue to accumulate, causing a large deviation from a true value.

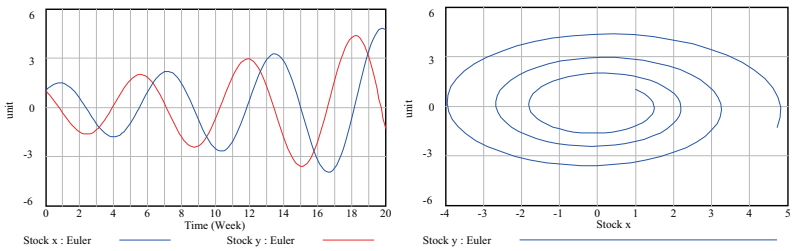


Figure 1.27: Oscillation under Euler Method

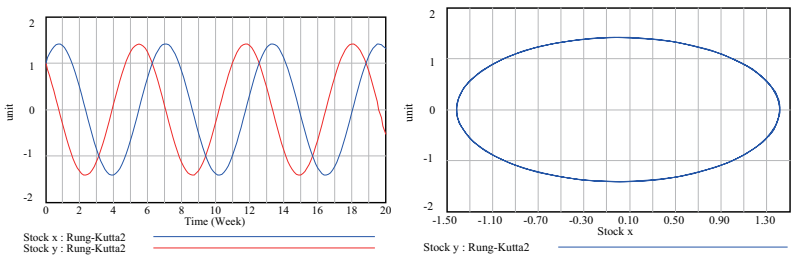


Figure 1.28: Oscillation under Runge-Kutta2 Method

On the other hand, the 2nd-order Runge-Kutta solution eliminates this deviation and yields a periodic or cyclical movement as illustrated in Figure 1.28. This gives us a caveat that setting a small number of dt in the Euler's method is not enough to approximate a true value in the case of continuous flow. It is expedient, therefore, to examine the computational results by both methods and see whether they are differentiated or not.

Examples of Oscillation

Pendulum movement is a typical example of oscillation. It is shown in [6] that employment instability behavior is produced by the same system structure which generates pendulum oscillation. Predator - prey model is another example of oscillation heavily used in ecology.

1.7 Delays in System Dynamics

1.7.1 Material Delays

First-Order Material Delays

System dynamics consists of four building blocks or letters: stock, flow, variable and arrow, as already discussed in the beginning of this chapter. To generate fundamental behaviors, these letters have to be combined according to its grammatical rules: reinforcing (positive) and balancing (negative) feedback loops, and delays. So far reinforcing and balancing feedback loops have been explored. Yet delays have been already applied in our models above without focusing on them. Delays play an important role in model building. Accordingly, it is appropriate to examine the meaning of delays in system dynamics in this section.

Delays in system dynamics has a structure illustrated in Figure 1.29. That is, output always gets delayed when input goes through stocks. This is an inevitable feature in system dynamics. It is essential in system dynamics to distinguish two types of delays: material delays and information delays. Let us start with material delays first.

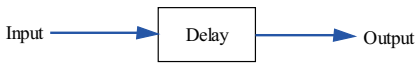


Figure 1.29: Structure of Delays

When there is only one stock, delays becomes similar to exponential decay for the one-time input, which is called *pulse*. In Figure 1.30, 100 units of material are input at time zero. This corresponds to the situation, for instance, in which 100 units of goods are purchased and stored in inventory, or 100 letters are dropped in the post office. Delay time is assumed to be 6 days in this example. In other words, one-sixth of goods are to be delivered daily as output.

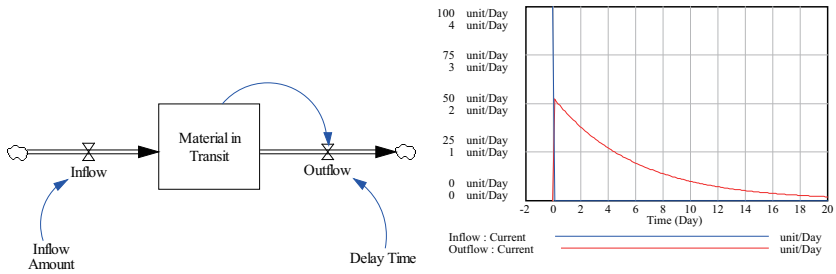


Figure 1.30: First-Order Material Delays

Second-Order Material Delays

In the second-order material delays, materials are processed twice as illustrated in the top diagram of Figure **??**. Total delay time is the same as 6 days. Accordingly, delay time for each process becomes 3 days. In this case, output distribution becomes bell-shaped. The reader can expand the delays to the n-th order and see what will happen to output.

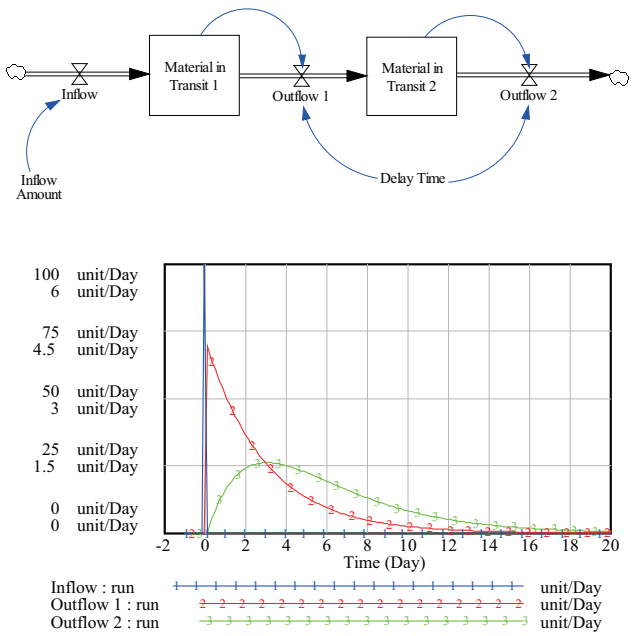


Figure 1.31: Second-Order Material Delays

1.7.2 Information Delays

First-Order Information Delays

Information delays occur because information as input has to be processed by human brains and implemented as action output. Information literally means in-form; that is, being input to brain which forms it for action. This is a process to adjust our perceived understanding in the brain to the actual situation outside the brain. Structurally this is the same as balancing feedback explained above to fill the gap between x^* and x, as illustrated in the left-hand diagram of Figure 1.32.

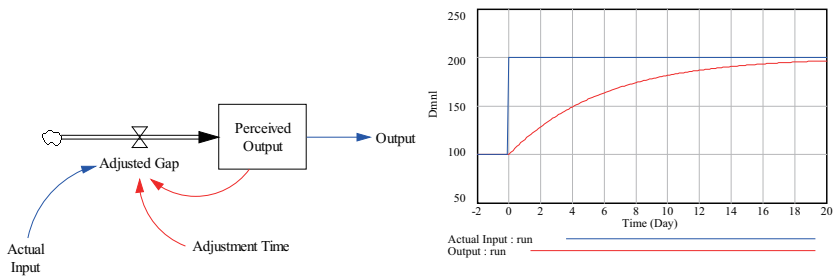
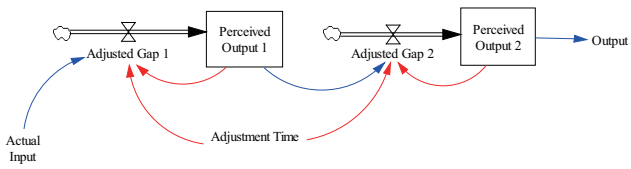


Figure 1.32: First-Order Information Delays

Our perceived understanding, say, on daily sales order, is assumed here to be 100 units, yet actual sales jumps to 200 at the time zero. Our suspicious brain hesitates to adjust to this new reality instantaneously. Instead, it slowly adapts to a new reality with the adjustment time of 6 days. This type of adjustment is explained as adaptive expectations and exponential smoothing in [72].

Second-Order Information Delays

Second-order information delays imply that information processing occurs through two brains. This is the same as re-thinking process for one person or a process in which information is being sent to another person. This structure is modeled in the top diagram of Figure 1.33. The adaptation process of the second-order information delays becomes slower than that of the first-order information delays as illustrated in the bottom diagram of the Figure.



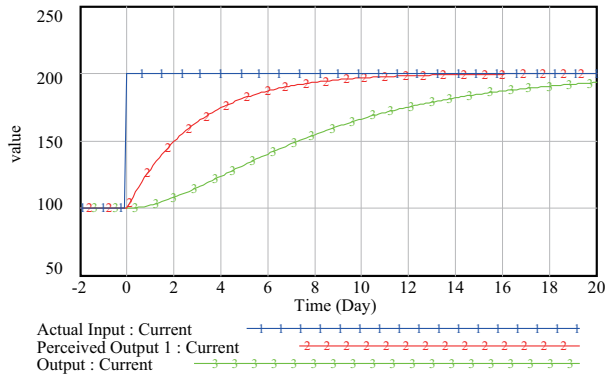


Figure 1.33: Second-Order Information Delays

Adaptive Expectations for Random Walk

First-order information delays are also called adaptive expectations or exponential smoothing because perceived output tries to adjust gradually to the actual input as illustrated in Figure 1.34.

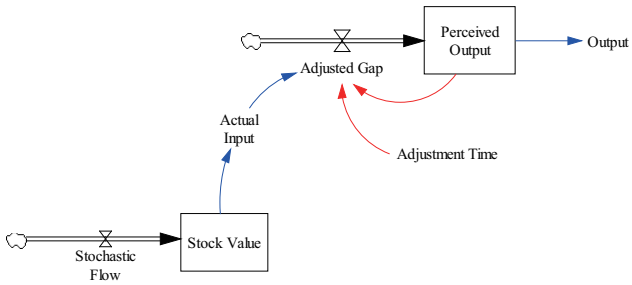


Figure 1.34: Adaptive Expectations Model for Random Walk

This exponentially smoothing behaviors are illustrated in Figure 1.35. That is, random walk becomes actual input (line 2), then its gap with the Perceived Output is adjusted (smoothed) by the first-order information delays, which becomes output (line 1).

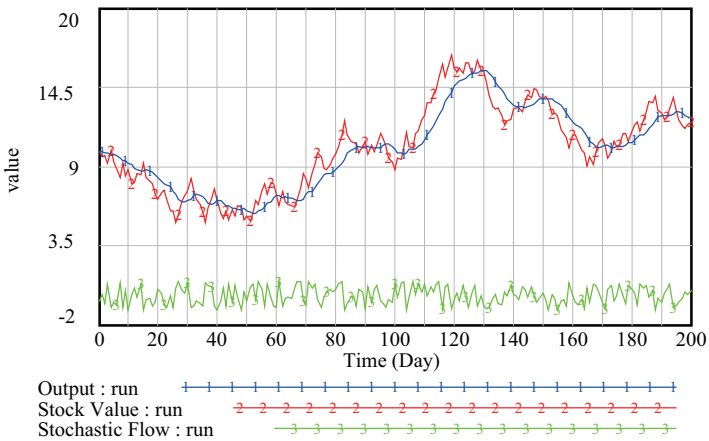


Figure 1.35: Adaptive Expectations Behaviors for Random Walk

1.8 System Dynamics with Three Stocks

1.8.1 Feedback Loops in General

It has been shown that system dynamics with two stocks can mostly produce all fundamental behavior patters such as exponential growth, exponential decay, S-shaped limit to growth, overshoot and collapse, and oscillation. Actual behaviors observed in complex system are combinations of these fundamental behaviors. We are now in a position to build system dynamics model based on these fundamental building blocks. And this introductory chapter on system dynamics seems appropriate to end at this point, and we should go to next chapter in which how system dynamics method can be applied to economics.

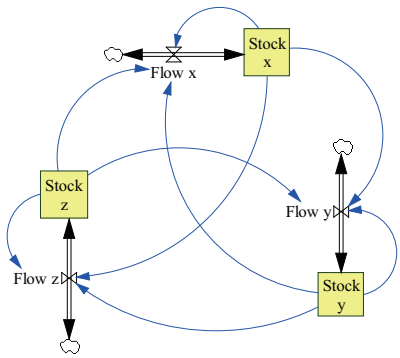


Figure 1.36: Feedback Loop in General

Yet, there exist another behaviors which can not be produced with two stocks; that is a chaotic behavior! Accordingly, we stay here for a while, and consider a general feedback relation for the case of three stock-flow relations. Figure 1.36 illustrates a general feedback loops.

Each stock-flow relation has its own feedback loop and two mutual feedback loops. In total, there are 6 feedback loops, excluding overlapping ones. As long as we observe *the parts* of mutual loop relations, that's all loops. However, if we observe *the whole*, we can find two more feedback loops: that is, a whole feedback loop of $x \to y \to z \to x$, and $x \to z \to y \to x$. Therefore, there are 8 feedback loops as a whole. The existence of these two whole feedback loops seems to me to symbolize a complex system in terms of loops; that is, *the whole is more than the sum of its parts.*

> A complex system is one whose component parts interact with sufficient intricacy that they cannot be predicted by standard linear equations; so many variables are at work in the system that its overall behavior can only be understood as an emergent consequence of the holistic sum of all the myriad behaviors embedded within. Reductionism does not work with complex systems, and it is now clear that a purely reductionist approach cannot be applied when studying life: in living systems, *the whole is more than the sum of its parts* (emphasis is made by the author) [50], pp. 7-8.

Mathematically, this general feedback loop relation can be represented by a following dynamical system in which each flow is a function of all stocks x, y and z.

$$\frac{dx}{dt} = f(x, y, z) \tag{1.59}$$

$$\frac{dy}{dt} = g(x, y, z) \tag{1.60}$$

$$\frac{dz}{dt} = h(x, y, z) \tag{1.61}$$

1.8.2 Lorenz Chaos

As a special example of the general feedback loops by three stock-flow relations, let us consider well-known Lorenz equations which yield a chaotic movement. Mathematical equations of the Lorenz chaos are written as

$$\frac{dx}{dt} = -a(x - y) \tag{1.62}$$

$$\frac{dy}{dt} = -xz + bx - y \tag{1.63}$$

$$\frac{dz}{dt} = xy - cz \tag{1.64}$$

Figure 1.37 illustrates feedback loops of the Lorenz equations. Compared with a general case of the above Figure 1.36, a link from the Stock z to the Flow x is missing. Accordingly, we have 6 feedback loops in total - a loss of two loops ! We are not sure if this loss of two loops is related with chaotic behaviors to be discussed below.

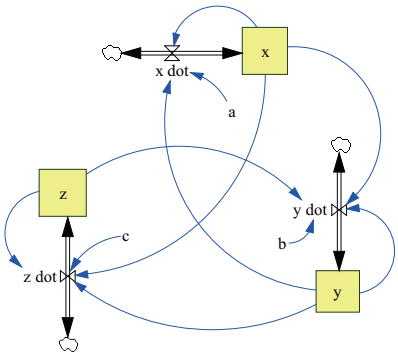


Figure 1.37: Lorenz Feedback Loop

Equilibrium of this Lorenz equations is obtained such that

$$x^* = \pm\sqrt{(b-1)c} \tag{1.65}$$

$$y^* = \pm\sqrt{(b-1)c} \tag{1.66}$$

$$z^* = b-1 \tag{1.67}$$

Parameter values of Lorenz equations are assigned as $a = 10,\ b = 28$ and $c = 8/3$. Hence, equilibrium values have to be $x^* = y^* = 3\sqrt{8}$ and $z^* = 27$. Initial values are instead set at off-equilibrium values such that $x(0) = 0,\ y(0) = 2$ and $z(0) = 0$. (See Chapter 14 The Lorenz System in [41]).

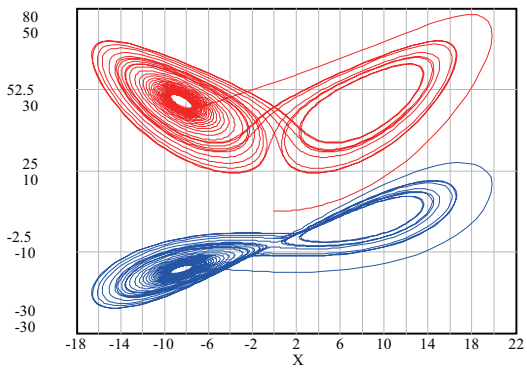


Figure 1.38: Lorenz Chaos

Figure 1.38 illustrates two phase diagrams of stocks x - y and x - z, that is, movements of the stock y and z are illustrated on the y-axis against the stock x on the x-axis. Apparently, their off-equilibrium behaviors fail to restore equilibrium, instead they begin to be attracted to a phase diagram, called *Lorenz*

attractor of chaos, wherever they initially start. Calculations in the Figure are done by the 4th-order Runge-Kutta method at $dt = 0.0078125$; that is, at each of 128 sub-periods in a unit period. With such a small sub-period, computational errors may arise less likely as explained above.

Sensitive Dependence on Initial Conditions

In the above Lorenz phase diagram, movement of stocks does not converge to a fixed point or a limit cycle, or diverge to infinity. Instead, wherever it starts, it seems to be eventually attracted to a certain region and continue fluctuating in it, with the information of its start being lost eventually. That region is called *a strange attractor* or *chaos*. One of the main features of chaos is a sensitive dependence on initial conditions. This is numerically explained as follows. Suppose a true initial value of the stock y in the Lorenz equations is $y^*(0) = 2.0001$ instead of $y(0) = 2.0$, and denote its true value by y^*. At the period $t = 20$ those two values of the stock are calculated as $y(t) = 16.1513$, and $y^*(t) = 16.1453$. The difference is only 0.006 and they stay very close each other. This makes sense, because both started at the very close distance of 0.0001. To our surprise, however, at the period, say, $t = 26.5$, they are calculated as $y(t) = -2.25909$, and $y^*(t) = 9.9033$; a large difference of 12.16239 is made. Small amount of differences at an initial time eventually turns out to cause a big difference later. In other words, stock values sensitively depend on their initial conditions. Figure 1.39 illustrates how values of the stock y (line 1) begin to diverge from a true value y^* (line 2) around the period $t = 25.6$.

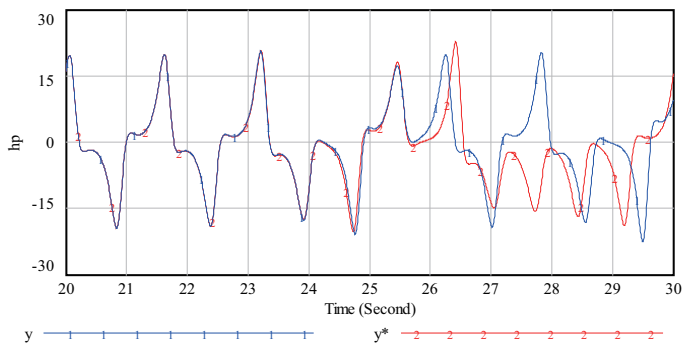


Figure 1.39: Sensitive Dependence on Initial Conditions

Why could it be possible? It is caused by the power of exponential magnification empowered by feedback loops. As illustrated in Figure 1.16, for instance, a simple calculation yields that an initial difference of 0.001 is exponentially magnified to 22.02 by the time $t = 100$, more than twenty-two thousand factors larger, because of a positive feedback loop. Chaos is a region called strange attractor to which infinitely many iterated and exponentially magnified values

are confined. Hence, it is intuitively understood that exponentially magnified values in a chaos region sensitively depend on initial conditions; in other words, values whose initial conditions differ only very slightly cannot stay close and begin to diverge eventually.

This chaotic feature creates annoying problem in system dynamics: unpredictability in the future. It is almost impossible in reality to obtain true initial values due to some observation errors and round-off errors of measurement and computations. These errors are magnified in a chaotic system dynamics to a point where predictions of the future and forecasting become almost meaningless and misleading. If analytical solutions of differential equations could be found, this would never happen, because solutions are continuous function of time and we could easily predict or approximate the future behavior of the system even if initial conditions are missed slightly. Without the analytical solutions, the future has to be iterated step by step, causing an exponential magnification by feedback loop. Unfortunately as discussed above, it is almost impossible to find analytical solutions in a nonlinear dynamics and system dynamics. In such cases, if a true initial value fails to be specified, then we cannot predict the future at all, even if we try to make calculations as precise as possible by employing Runge-Kutta methods and making sub-periods smaller as discussed in the previous sections. Hence, system dynamics becomes inefficacious as a forecasting simulation method.

What's a good use of system dynamics, then? If a dynamic system is chaotic, all values of stocks are attracted to a region of strange attractor; in other words, information of initial conditions will be lost eventually and only patterns or structures of the system begin to reveal themselves. In system dynamics, these patterns and structures help us learn the behavior of the system we want to explore. System dynamics is a very effective learning method in that direction, not in the direction of futures prediction.

1.9 Chaos in Discrete Time

1.9.1 Logistic Chaos

Famous logistic function which produces chaos is the following:

$$x_{t+1} = ax_t(1 - x_t) \qquad t = 0, 1, 2, 3, \cdots \tag{1.68}$$

(See Chapter 15 Discrete Dynamical Systems in [41])

To fit into our system dynamics presentation, its flow can be rewritten as follows:

$$\boxed{f(t) = ax_t(1 - x_t) - x_t} \tag{1.69}$$

As coefficient a increases stock x produces n-period oscillations $n = 2, 4, 8, \cdots$, and eventually produces chaos. Left-hand diagram of Figure 1.40 shows an chaotic movement for $a = 3.95$ and right-hand diagram is its phase diagram.

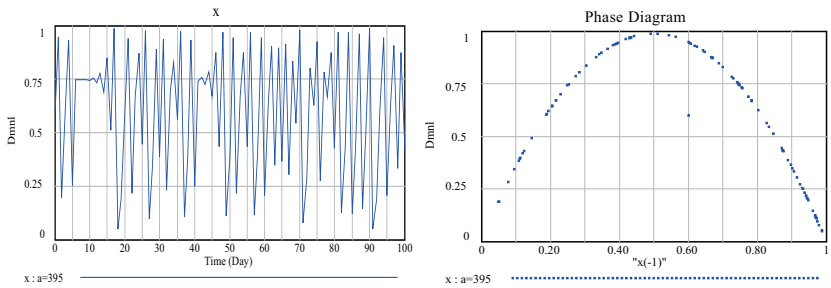


Figure 1.40: Chaos in Logistic Function

Chaotic movements do not fill in all space but fit into parabola shape as strange attractor.

This type of chaos in discrete time gradually disappears when DT(Delta Time) in simulation becomes smaller.

1.9.2 Discrete Chaos in S-shaped Limit to Growth

The equation (1.55) in the S-shaped limit to growth can be rewritten in a discrete format as

$$y_{t+1} = y_t + ay_t(b - y_t) \qquad t = 0, 1, 2, 3, \cdots \tag{1.70}$$

In other words, flow becomes

$$\boxed{f(t) = ay_t(b - y_t)}, \tag{1.71}$$

which also becomes the same as the right-hand side of the equation (1.68) for $b = 1$.

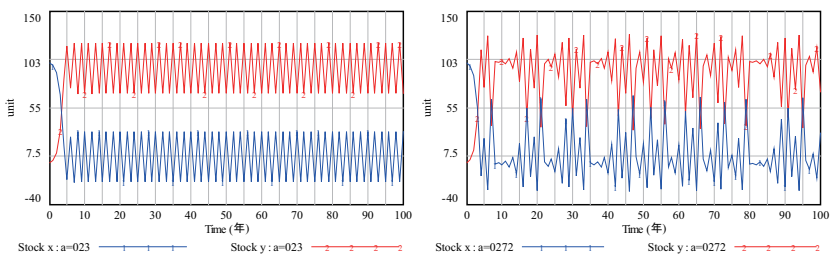


Figure 1.41: Chaos in S-shaped Limit to Growth

To our surprise, it turns out that the flow (1.71) can also produce chaos if $f(t)$ is allowed to take negative values. In other words, S-shaped limit to growth

behavior turns out to be chaotic if flows move forward and backward between stocks x and y. Figure 1.41 illustrates such chaotic behaviors. Specifically, its left-hand diagram shows a two-period cycle of stocks x and y for $a = 0.023$ and its right-hand diagram shows chaotic movements for $a = 0.0272$.

Appendix: Runge-Kutta Methods in General

Flow in equation (1.29) can be more generally described as a function of time and stock; that is,

$$\frac{dx}{dt} = f(t, x) \tag{1.72}$$

Accordingly, the Runge-Kutta methods need to be more generally formulated as follows[5]

2nd-Order Runge-Kutta Method

$$\begin{aligned} dx_1 &= f(t, x)dt && (1.73)\\ dx_2 &= f(t + dt, x + dx_1)dt && (1.74)\\ dx &= \frac{dx_1 + dx_2}{2} \\ &= \frac{f(t, x) + f(t + dt, x + dx_1)}{2}dt && (1.75) \end{aligned}$$

4th-Order Runge-Kutta Method

$$\begin{aligned} dx_1 &= f(t, x)dt && (1.76)\\ dx_2 &= f(t + \frac{dt}{2}, x + \frac{dx_1}{2})dt && (1.77)\\ dx_3 &= f(t + \frac{dt}{2}, x + \frac{dx_2}{2})dt && (1.78)\\ dx_4 &= f(t + dt, x + dx_3)dt && (1.79)\\ dx &= \frac{dx_1 + 2dx_2 + 2dx_3 + dx_4}{6} && (1.80) \end{aligned}$$

Compared with the Euler's methods, the 2nd-order Runge-Kutta method requires twice as many calculations and the 4th-order Runge-Kutta method requires 4 times as many calculations. In other words, the number of calculation of the Euler's method for $dt = \frac{1}{4}$ is the same as the 2nd-order Runge-Kutta method for $dt = \frac{1}{2}$, which is also the same as the 4th-order Runge-Kutta method for $dt = 1$. Table 1.8 shows a combination of dt and three methods that induces the same number of calculations. Even so, from the results in Table 1.6, it can be easily verified that the Runge-Kutta methods produce better approximations for the same number of calculations.

Table 1.8: Same Number of Calculation

	dt	dt	dt
Euler	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
Runge-Kuttta 2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
Runge-Kutta 4	1	$\frac{1}{2}$	$\frac{1}{4}$

[5]Technical Documentation, STELLA Software, High Performance Systems, Inc. pp. 13-6, 13-7, 1997.